Proof that EM Drive Thrust/Power and Q scale as $\sqrt{L}$

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I prove that the thrust force per input power (for all three EM-Drive theories) scales like the square root of any geometrical dimension, for constant resistivity and magnetic permeability of the interior wall of the cavity and for constant geometrical ratios, constant medium properties and for the same mode shape. To maximize the thrust per input power, according to all three theories the most efficient EM-Drive would be as large as possible, this being due to the fact that the quality of factor of resonance Q (all else being equal) scales like the square root of the geometrical dimensions. Small cavity EM-Drives (all else being equal) are predicted to have smaller quality of resonance Q and therefore smaller thrust force/input power.

1. Thrust per power of EM Drive compared to a photon rocket

Here I briefly describe the thrust per power input claimed by various authors for the EM-Drive and its comparison to the one of a photon rocket. I start with the definition of Power $P(t)$ as the time derivative of work W, and therefore equal to the vector dot product of force times velocity,

$$P(t) = \frac{dW}{dt} = \vec{F} \cdot \vec{v} \tag{1}$$

For an ideal photon rocket with a perfectly collimated photon beam, the exhaust velocity (not the spaceship velocity!) is the speed of light c and therefore, $Fc = P_{in}$, where P_{in} is the power input into the exhaust ("power input" here only stands for the power at this late stage, notice that there may be further losses at earlier stages from the power plant, etc.). Therefore, for an ideal photon rocket, the "thrust" force per input power is,

$$\left(\frac{F}{P_{in}}\right)_{\text{photonRocket}} = \frac{1}{c} \tag{2}$$

Furthermore: For rockets exhausting particles-with-mass at speeds much lower than the speed of light, for example ion thrusters, this ratio is $2/v$ instead of $1/c$, where v is the speed of the particle-having-mass (as propellant). Particles-with-mass, unlike photons, need to be accelerated to the exhaust speed. The reason for the factor of 2 is because kinetic energy of a massive low speed particle is $E = (1/2)mv^2$ instead of the energy of a photon $E = mc^2$. Therefore, the efficiency (F/P_{in}) for ion thrusters is much larger than the one for photon rockets since $v \ll c$, and hence $2/v \gg 1/c$ and that is why this type of photon rocket has not seen, and is not envisioned to have, practical use.

Interestingly the "thrust" force per input power for the EM Drive, according to all three different theories (McCulloch, Shawyer and "*Notsosureofit*") can be expressed similarly as:

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$$\left(\frac{F}{P_{in}}\right)_{\text{EM-Drive}} = \frac{Q\,g}{c} \tag{3}$$

where Q is the quality factor of resonance (an inverse measure of damping) and g is a dimensionless factor due to geometry, relative magnetic permeability, relative electric permittivity and mode shape. The specific form of g depends on the specific theory of each author. So, the force per input power for an EM Drive is predicted to be superior to a photon rocket as follows:

$$\left(\frac{F}{P_{in}}\right)_{\text{EM-Drive}} / \left(\frac{F}{P_{in}}\right)_{\text{photonRocket}} = Q\,g \tag{4}$$

In other words, the theoretical outperformance of the EM-Drive is predicted to be due to just the quality of resonance Q and the dimensionless factor g. For the purpose of this discussion I will avoid dealing with the strange consequences of these theories regarding conservation of momentum and conservation of energy issues inherent to the concept of proposing a closed resonant electromagnetic cavity for space propulsion.

2. The specific form of the factor g for different theories

McCulloch [1], has presented a number of simple formulas for the EM-Drive, all having the general form as Eq.(3) above. The simplest of which has the following definition for the dimensionless factor g,

$$g_{McCulloch} = \left(\frac{L}{D_s} - \frac{L}{D_b}\right) \tag{5}$$

where L is the length of the truncated cone, measured perpendicular to the end faces, along the axis of axial symmetry of the cone. D_s is the diameter of the small end of the truncated cone and D_b is the diameter of the big end of the truncated cone. So, it is evident that for this formula from McCulloch, the factor g is a dimensionless factor that only depends on the geometrical ratios L/D_s and L/D_b. It is also obvious that if one scales the EM-Drive geometry, such that the geometrical ratios L/D_s and L/D_b are kept constant, that the dimensionless factor g will remain constant in McCulloch's equation.

Shawyer [2], has presented a formula for the EM-Drive where the dimensionless factor g is defined as follows: $g_{Shawyer} = 2D_f$ where D_f is a dimensionless factor called the *Design Factor* by Shawyer, and where D_f is a function of the diameter-to-length ratios and in addition is also a function of the relative magnetic permeability $\mu_{r_{medium}}$ and the relative electric permittivity $\varepsilon_{r_{medium}}$, as well as the natural frequency of resonance and its associated mode shape (with associated mode shape numbers m, n, p),

$$g_{Shawyer} = g_{Shawyer}\left(\frac{L}{D_s}, \frac{L}{D_b}, \mu_{r_{medium}}, \varepsilon_{r_{medium}}, m, n, p\right) \tag{6}$$

where the diameters of the truncated cone appear explicitly in his formula for the design factor and where the length and the mode shape numbers appear only implicitly because the design factor is dependent on the natural frequency at which resonance with a particular mode shape occurs. It is simple to show that if one scales the EM-Drive geometry such that the geometrical ratios L/D_s and L/D_b, and the medium properties $\mu_{r_{medium}}, \varepsilon_{r_{medium}}$ are kept constant, and the mode shape is kept the same, that the dimensionless factor g will remain constant in Shawyer's equation.

Notsosureofit [3], has presented a more sophisticated formula for the EM-Drive, with explicit dependence on the mode shape, where the dimensionless factor g is defined as follows,

$$g_{Notsosureofit} = \left(\frac{\psi_{mn}^2}{4\pi^3}\right)\left(\frac{c}{f_{mnp}}\right)^3 \frac{1}{L}\left(\frac{1}{D_s^2} - \frac{1}{D_b^2}\right) \tag{7}$$

where $\psi_{mn} = x_{mn}$ (the n^{th} zeros of the cylindrical Bessel function (of the first kind) $J_m(x)$) for transverse magnetic (TM) modes, and $\psi_{mn} = x'_{mn}$ (the n^{th} zeros of the first derivative $J'_m(x)$ of the cylindrical Bessel functions (of the first kind) $J_m(x)$) for transverse electric (TE) modes.

Side note: This link [4], is an excellent source for the numerical values of the n^{th} roots x_{mn} and x'_{mn} of $J_m(x)$ and $J'_m(x)$, respectively, for the following values of m and n: $m < 11$ and $n < 6$.

Therefore, it can be shown that the g factor in *Notsosureofit's* hypothesis is a function of the geometrical ratios, the medium properties and the mode shape of resonance:

$$g_{Notsosureofit} = g_{Notsosureofit}\left(\frac{L}{D_s}, \frac{L}{D_b}, \mu_{r_{medium}}, \varepsilon_{r_{medium}}, m, n, p\right) \tag{8}$$

which is the same form of nondimensional dependence as in Shawyer's $g_{Shawyer}$. Exactly how this is so will be shown in detail in the next section.

3. Natural frequency scaling

For simplicity, since the truncated cone resonant cavities tested by NASA, Shawyer, Tajmar, and others have all been close to a cylindrical cavity, I will derive the scaling relationship for the natural frequencies of a cylindrical cavity, but this can also be done with the more complicated equations for a truncated conical cavity. For an electromagnetically resonant cylindrical cavity the functions are: the cosine of the longitudinal coordinate z, the cosine of the cylindrical polar angular coordinate ϑ and the cylindrical Bessel functions $J_m(\kappa_{mn}\,\varrho)$ of the cylindrical polar radial coordinate ϱ (where $\kappa_{mn} = \frac{\psi_{mn}}{R}$ is the angular wave number associated with the circular cross-section of the cylinder, which for $p \neq 0$, in other words, for mode shapes with electromagnetic field not constant in the axial direction z, is different from the angular wave number $k_{mnp} = \omega_{mnp}\sqrt{\mu_{r_{medium}}\varepsilon_{r_{medium}}}/c$ for the cylindrical cavity). For an electromagnetically resonant truncated conical cavity instead, the functional dependence is expressed in terms of cosine functions in the azimuthal angle direction ϕ, associated Legendre functions P_n^m in the spherical polar angle (also called zenith angle) direction θ, and spherical Bessel functions $\frac{1}{\sqrt{r}}J_{\pm(n+1/2)}(k_{mnp}r)$ ([5] and[6]). (Here $k_{mnp} = \omega_{mnp}\sqrt{\mu_{r_{medium}}\varepsilon_{r_{medium}}}/c$ is the angular wave number and r is the spherical radial coordinate directed along the generatrix, which for zero spherical polar angle θ

coincides with the longitudinal axis z of symmetry of the cone, which is perpendicular to the direction of the radial polar coordinate ϱ for a cylinder. Hence it is important to distinguish between the spherical radial coordinate r and the cylindrical polar radial coordinate ϱ directions: they are very different directions. Also notice that the cylindrical Bessel function $J_m(\kappa_{mn}\,\varrho)$ for the cylinder is only associated with mode shape numbers m and n of the circular cross-sections perpendicular to the longitudinal axis z of the cylinder, and hence independent of mode shape number p, while the spherical Bessel function $\frac{1}{\sqrt{r}}J_{\pm(n+1/2)}(k_{mnp}r)$ for the truncated cone with spherical ends is associated with all mode shape numbers m, n and p of the entire truncated cone, including the trapezium shaped plane sections perpendicular to the azimuthal direction ϕ of the truncated cone).

The reason why all EM-Drive experiments have been performed up to now with EM-Drive geometries close to a cylindrical cavity is because experimenters have tried to follow Shawyer's prescription that, for a given frequency and mode shape, the small diameter of the truncated conical cavity should be larger than the diameter of an open cylindrical waveguide at the cut-off frequency for that mode shape (although the EM-Drive is a closed cavity, and not an open waveguide, and it is known that cut-off does not take place in truncated conical cavities under the same conditions). For practical applications to cavities resonating at the desired mode shapes: TE012 and TE013, this prescription forbids geometries of truncated cones where the small diameter is much different from the big diameter. Therefore it turns out that one can use a mean radius, $R = (D_s + D_b)/4$ to model the truncated cone as a cylindrical cavity, having natural frequencies f_{mnp}

$$f_{mnp} = \frac{c}{R} a_{mnp} \tag{9}$$

angular wave number $k_{mnp} = 2\,\pi\, a_{mnp}\sqrt{\mu_{r_{medium}}\varepsilon_{r_{medium}}}/R$ (radians per unit length), wavelength $\lambda_{mnp} = R/\left(a_{mnp}\sqrt{\mu_{r_{medium}}\varepsilon_{r_{medium}}}\right)$, and where c is the speed of light, R is the previously defined mean radius and where m, n, p are the so called "mode shape numbers" defining the mode shape, where for a cylinder, m is the integer related to the circumferential direction (cylindrical polar angle ϑ direction), n is the integer related to the cylindrical polar radial direction (ϱ direction) and p is the integer related to the longitudinal axial direction (z cylindrical polar axis). From the closed-form solution for an electromagnetically resonant cylindrical cavity (for example Eq.(7.56) of Collin[7], or Eqs.(9.39a) and (9.45) of Balanis[8]) it follows that:

$$a_{mnp} = \sqrt{\frac{(\psi_{mn}/\pi)^2 + (p\,R/L)^2}{4\mu_{r_{medium}}\varepsilon_{r_{medium}}}} \tag{10}$$

It is also trivial to show that since the mean radius is $R = (D_s + D_b)/4$ then the ratio of the mean radius to the length can be expressed in terms of the geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$:

$$\begin{aligned} \frac{R}{L} &= \frac{1}{4}\left(\frac{D_s}{L} + \frac{D_b}{L}\right) \quad \text{hence} \\ a_{mnp} &= a_{mnp}\left(\frac{L}{D_s}, \frac{L}{D_b}, \mu_{r_{medium}}, \varepsilon_{r_{medium}}, m, n, p\right) \end{aligned} \tag{11}$$

for constant geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$, constant medium properties $\mu_{r_{medium}}$, $\varepsilon_{r_{medium}}$, and for the same mode shape m, n, p, a_{mnp} will remain constant. Since the frequency scales like $\frac{c}{R}$, and R divided by L, or D_s, or D_b can be expressed in terms of the geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$, it follows that the frequency f_{mnp} scales like the inverse of any geometrical dimension $\frac{c}{L}$, $\frac{c}{D_s}$ or $\frac{c}{D_b}$ and the geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$, the medium properties and the mode shape:

$$\begin{aligned} f_{mnp} &= \frac{c}{R} a_{mnp}\left(\frac{L}{D_s}, \frac{L}{D_b}, \mu_{r_{medium}}, \varepsilon_{r_{medium}}, m, n, p\right) \\ &= f_{mnp}\left(\frac{c}{L}, \frac{L}{D_s}, \frac{L}{D_b}, \mu_{r_{medium}}, \varepsilon_{r_{medium}}, m, n, p\right) \\ &= f_{mnp}\left(\frac{c}{D_s}, \frac{L}{D_s}, \frac{L}{D_b}, \mu_{r_{medium}}, \varepsilon_{r_{medium}}, m, n, p\right) \\ &= f_{mnp}\left(\frac{c}{D_b}, \frac{L}{D_s}, \frac{L}{D_b}, \mu_{r_{medium}}, \varepsilon_{r_{medium}}, m, n, p\right) \end{aligned} \tag{12}$$

To illustrate this for *Notsosureofit's* dimensionless factor, substituting Eq. (9) into Eq. (7) it follows that:

$$g_{Notsosureofit} = \left(\frac{\psi_{mn}^2}{4\pi^3}\right)\left(\frac{1}{a_{mnp}}\right)^3 \frac{R}{L}\left(\left(\frac{R}{D_s}\right)^2 - \left(\frac{R}{D_b}\right)^2\right) \tag{13}$$

therefore the dimensionless factor $g_{Notsosureofit}$ depends on the ratio of the mean radius R to the length L and on the square of the ratio of the mean radius R to the diameters D_s and D_b. Since the ratio of the mean radius R to the length L or to the diameters D_s, D_b of the EM-Drive can be expressed in terms of the geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$:

$$\begin{aligned} \frac{R}{L} &= \frac{1}{4}\left(\frac{D_s}{L} + \frac{D_b}{L}\right) \\ \left(\frac{R}{D_s}\right)^2 &= \frac{1}{16}\left(1 + \frac{\frac{L}{D_s}}{\frac{L}{D_b}}\right)^2 \\ \left(\frac{R}{D_b}\right)^2 &= \frac{1}{16}\left(1 + \frac{\frac{L}{D_b}}{\frac{L}{D_s}}\right)^2 \end{aligned} \tag{14}$$

then it follows that the g factor in *Notsosureofit's* hypothesis is a function of the geometrical ratios, the medium properties and the mode shape of resonance:

$$g_{Notsosureofit} = g_{Notsosureofit}\left(\frac{L}{D_s}, \frac{L}{D_b}, \mu_{r_{medium}}, \varepsilon_{r_{medium}}, m, n, p\right) \tag{15}$$

Therefore for constant geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$, constant medium properties $\mu_{r_{medium}}$, $\varepsilon_{r_{medium}}$, and for the same mode shape m, n, p, the dimensionless factor g will remain constant. It is trivial to show the same result for Shawyer's design factor, and hence for the dimensionless factor g in Shawyer's expression. So, in general I can state that all theoretical expressions, McCulloch's, Shawyer's and *Notsosureofit's*, are such that the dimensionless factor g will remain constant for constant geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$, constant medium properties $\mu_{r_{medium}}$, $\varepsilon_{r_{medium}}$, and for the same mode shape m, n, p.

4. Quality of resonance (Q) scaling

The quality of resonance factor (Q) is defined as follows:

$$\begin{aligned} Q &\stackrel{\text{def}}{=} 2\pi \frac{\text{EnergyStored}}{\text{EnergyDissipatedPerCycle}} \\ &\stackrel{\text{def}}{=} \omega_{mnp} \frac{\text{EnergyStored}}{\text{PowerLoss}} \end{aligned} \tag{16}$$

where:

$$\begin{aligned} \omega_{mnp} &= \text{resonant angular frequency} \\ &= 2\pi f_{mnp} \\ f_{mnp} &= \text{resonant frequency with mode shape numbers } m, n, p \\ \text{EnergyStored } &= \int \text{ ElectromagneticEnergyDensity } dV \\ \text{PowerLoss } &= \frac{\omega_{mnp}\delta}{2} \int \text{ ElectromagneticEnergyDensity } dA \\ &= \frac{R_s}{\mu_{wall}} \int \text{ ElectromagneticEnergyDensity } dA \\ &= \frac{\rho}{\mu_{wall}\delta} \int \text{ ElectromagneticEnergyDensity } dA \\ R_s &= \text{ surface resistance} \\ &= \frac{\rho}{\delta} \\ \rho &= \text{ resistivity of the interior wall of the EM Drive resonant cavity} \\ \mu_{wall} &= \text{ magnetic permeability of the interior wall of EM Drive} \\ &= \mu_0 \mu_{r_{wall}} \\ \delta &= \text{ skin depth (penetration depth of the electromagnetic energy)} \\ V &= \text{ interior volume of EM Drive resonant cavity} \\ A &= \text{ interior surface of EM Drive resonant cavity} \end{aligned}$$

In general, for arbitrary frequencies, the skin depth is:

$$\delta = \sqrt{\frac{2\rho}{\omega\mu_{wall}} \left(\sqrt{1 + (\rho\omega\epsilon_{wall})^2} + \rho\omega\varepsilon_{wall} \right)} \tag{17}$$

where $\varepsilon_{wall} = \varepsilon_0 \varepsilon_{r_{wall}}$ = electric permittivity of the interior wall of the EM-Drive resonant cavity. At angular frequencies ω much below $1/(\rho\varepsilon_{wall})$, for example, in the case of copper, for frequencies much below exahertz (10^9 GHz, the range of hard X-rays and Gamma rays), the skin depth can be expressed as follows,

$$\delta = \sqrt{\frac{2\rho}{\omega\mu_{wall}}} \tag{18}$$

Now, at resonance $\omega = \omega_{mnp}$, using the fact that

$$\text{PowerLoss } = \frac{\omega_{mnp}\delta}{2} \int \text{ ElectromagneticEnergyDensity } dA$$

substituting into Eq. (16) definition for the quality factor of resonance, one immediately obtains,

$$Q = \frac{2}{\delta} \frac{\int \text{ Electromagnetic Energy Density } dV}{\int \text{ Electromagnetic Energy Density } dA} \tag{19}$$

Alternatively one can arrive at the same result, using the formula for power loss that depends on the *surface resistance* R_s,

$$\begin{aligned} \text{PowerLoss} &= \frac{R_s}{\mu_{wall}} \int \text{ ElectromagneticEnergyDensity } dA \\ &= \frac{\rho}{\mu_{wall}\delta} \int \text{ ElectromagneticEnergyDensity } dA \end{aligned}$$

and substituting this into the definition for the quality factor of resonance Eq. (16), one gets,

$$\begin{aligned} Q &= \frac{\omega_{mnp}\mu_{wall}}{R_s} \frac{\int \text{ Electromagnetic Energy Density } dV}{\int \text{ Electromagnetic Energy Density } dA} \\ &= \frac{\omega_{mnp}\mu_{wall}\delta}{\rho} \frac{\int \text{ Electromagnetic Energy Density } dV}{\int \text{ Electromagnetic Energy Density } dA} \end{aligned} \tag{20}$$

and using the fact that at angular frequencies ω much lower than $1/(\rho\varepsilon)$ the angular frequency ω is a function of the square of the skin depth δ,

$$\omega = \frac{2\rho}{\mu_{wall}\delta^2} \tag{21}$$

it is straightforward to show that the quality of resonance Q is:

$$Q = \frac{2}{\delta} \frac{\int \text{ Electromagnetic Energy Density } dV}{\int \text{ Electromagnetic Energy Density } dA} \tag{22}$$

the electromagnetic energy density integrated over the cavity volume, divided by the electromagnetic energy density integrated over the cavity surface area, divided by the skin depth.

Skin depth scaling: At frequencies much below $1/(\rho\varepsilon)$ the skin depth at a resonant frequency f_{mnp} can be expressed as

$$\delta = \sqrt{\frac{\rho}{\mu_{wall}\pi f_{mnp}}} \tag{23}$$

Substituting the expression for frequency Eq. (9), $f_{mnp} = \frac{c}{R} a_{mnp}$, into the above skin depth equation, results in the following expression:

$$\delta = \sqrt{R}\sqrt{\frac{\rho}{\mu_{wall}\,\pi\, c\, a_{mnp}}} \tag{24}$$

Using the previously derived expression for a_{mnp} Eq. 11 and Eq. 14 for the dimensional ratios, one concludes that the skin depth δ scales like the square root of any geometrical dimension, for constant resistivity ρ and magnetic permeability μ_{wall} of the interior wall of the cavity, for constant geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$, constant medium properties $\mu_{r_{medium}}, \varepsilon_{r_{medium}}$ and for the same mode shape m, n, p. In other words, for increasing dimensions of the cavity, preserving all geometrical ratios, and keeping medium properties constant and for the same mode shape, the skin depth will increase with the square root of the dimension, while the frequency will decrease, as the inverse of the dimension.

Quality of resonance (Q) scaling: Having determined the scaling law for the skin depth, what now remains to be shown is the scaling for the energy integral ratio in the expression for Q,

$$Q = \frac{2}{\delta}\left(\frac{\int \text{Electromagnetic (EM) Energy Density } dV}{\int \text{ Electromagnetic (EM) Energy Density } dA}\right) \tag{25}$$

The expressions under the integrals are dependent on each mode shape, as the electromagnetic energy distribution depends on mode shape, of course. However, notice that the lowest mode shapes (those with low values of mode shape numbers m, n, p, for example TE012, TE013, TM212) have been of interest in the EM Drive experiments so far. So, for simplification purposes assume that the distribution of the electromagnetic field is of low order, and hence not that much variable throughout the cavity, for low m, n, p number mode shapes (for example m=0, associated with the mode shape numbers TE012 and TE013 used by Shawyer, means a constant distribution in the azimuthal circumferential direction of the cavity). Under this assumption one can (for approximation purposes) take the energy density out of the volume and surface integrals:

$$\begin{aligned}\left(\frac{\int \text{ EM Energy Density } dV}{\int \text{ EM Energy Density } dA}\right) \quad &\sim \quad \left(\frac{\text{EM Energy Density}}{\text{EM Energy Density}}\right)\left(\frac{\int dV}{\int dA}\right) \\ &\sim \quad \frac{\text{InteriorVolume}}{\text{InteriorSurfaceArea}} \\ &\sim \quad \frac{\pi R^2 L}{2\pi R(R+L)} \\ &\sim \quad \frac{R}{2(1+R/L)}\end{aligned} \tag{26}$$

and substituting this and the previously found scaling law for the skin depth, into the expression for the quality of resonance factor Q, leads to:

$$
\begin{aligned}
Q &= \frac{2}{\delta}\left(\frac{\int \text{ EM Energy Density } dV}{\int \text{ EM Energy Density } dA}\right) \qquad (27)\\
&\sim \frac{2}{\sqrt{R}\sqrt{\rho/(\mu_{wall}\,\pi\, c\, a_{mnp})}}\,\frac{R}{2(1+R/L)}\\
&\sim \sqrt{R}\frac{1}{(1+R/L)}\sqrt{\frac{\mu_{wall}\,\pi\, c\, a_{mnp}}{\rho}}\\
&\sim \sqrt{L}\frac{\sqrt{\frac{D_s}{L}+\frac{D_b}{L}}}{2(1+\frac{1}{4}\left(\frac{D_s}{L}+\frac{D_b}{L}\right))}\sqrt{\frac{\mu_{wall}\,\pi\, c\, a_{mnp}}{\rho}}\\
&\sim \sqrt{D_s}\frac{\sqrt{1+\frac{\frac{L}{D_s}}{\frac{L}{D_b}}}}{2(1+\frac{1}{4}\left(\frac{D_s}{L}+\frac{D_b}{L}\right))}\sqrt{\frac{\mu_{wall}\,\pi\, c\, a_{mnp}}{\rho}}\\
&\sim \sqrt{D_b}\frac{\sqrt{1+\frac{\frac{L}{D_b}}{\frac{L}{D_s}}}}{2(1+\frac{1}{4}\left(\frac{D_s}{L}+\frac{D_b}{L}\right))}\sqrt{\frac{\mu_{wall}\,\pi\, c\, a_{mnp}}{\rho}}
\end{aligned}
$$

where the dimensionless mode shape factor a_{mnp} is:

$$
\begin{aligned}
a_{mnp} &= \sqrt{\frac{(\psi_{mn}/\pi)^2+(p\,R/L)^2}{4\mu_{r_{medium}}\varepsilon_{r_{medium}}}}\\
&= \sqrt{\frac{(\psi_{mn}/\pi)^2+(\frac{p}{4}\left(\frac{D_s}{L}+\frac{D_b}{L}\right))^2}{4\mu_{r_{medium}}\varepsilon_{r_{medium}}}}
\end{aligned}
$$

Therefore one concludes that the quality of resonance Q scales like the square root of any geometrical dimension L, D_s or D_b, for constant resistivity ρ and magnetic permeability μ_{wall} of the interior wall of the cavity and for constant geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$, constant medium properties $\mu_{r_{medium}}, \varepsilon_{r_{medium}}$, and for the same mode shape m, n, p. In other words, for increasing dimensions of the cavity, preserving all geometrical ratios, keeping medium properties constant and for the same mode shape, the quality of resonance Q will increase with the square root of the dimension, also the skin depth will increase with the square root of the dimension, while the frequency will decrease, as the inverse of the dimension.

Furthermore, I previously proved that all three theories for the EM Drive (McCulloch, Shawyer and *Notsosureofit*) have expressions for the force/input power proportional to the quality of factor Q times a dimensionless factor g,

$$
\begin{aligned}
\left(\frac{F}{P_{in}}\right)_{\text{EM-Drive}} \Big/ \left(\frac{F}{P_{in}}\right)_{\text{photonRocket}} &= Q\,g\\
\left(\frac{F}{P_{in}}\right)_{\text{EM-Drive}} &= \frac{Q\,g}{c} \qquad (28)
\end{aligned}
$$

and I previously proved that the dimensionless factor g (for all three theories: McCulloch, Shawyer and *Notsosureofit*) remains perfectly constant for constant geometrical ratios, constant medium properties $\mu_{r_{medium}}, \varepsilon_{r_{medium}}$ and for the same mode shape m, n, p. Therefore one concludes that the force per input power (for all three theories: McCulloch, Shawyer and *Notsosureofit*) scales like the square root of any geometrical dimension, for constant resistivity ρ and magnetic permeability μ_{wall} of the interior wall of the cavity and for constant geometrical ratios $\frac{L}{D_s}$, $\frac{L}{D_b}$, constant medium properties $\mu_{r_{medium}}, \varepsilon_{r_{medium}}$ and for the same mode shape m, n, p.

In other words, to maximize the force per input power, according to all three theories: (McCulloch, Shawyer and *Notsosureofit*) the most efficient EM-Drive would be as large as possible, this being due to the fact that the quality of factor of resonance Q (all else being equal) scales like the square root of the geometrical dimensions. Small cavity EM-Drives (all else being equal) are predicted to have smaller quality of resonance Q and therefore smaller thrust force/input power.

It is not clear whether this has been known to EM-Drive experimenters, given the fact that the recent experiments by Prof. Tajmar at TU Dresden, Germany, (under advice from Roger Shawyer according to the report [9]) were performed with a much smaller EM-Drive than previously tested by Shawyer and by NASA [10], and the fact that there are several EM-Drive researchers discussing really tiny EM-Drives (as the group in Aachen, Germany [11]) for use in CubeSats. Such EM Drives are predicted to be less efficient, having lower thrust force/input power, if the claimed thrust is not an experimental artifact.

5. Numerical verification analysis

The scaling law for the EM-Drive discussed in the previous sections is verified numerically using the exact solution for a truncated cone in terms of spherical Bessel and associated Legendre functions, using Wolfram *Mathematica*, and the experimental results from NASA [10].

NASA's truncated cone dimensions and material

$D_b = 11.01$ inch $= 0.279654$ m
$D_s = 6.25$ inch $= 0.15875$ m
$L = 9.00$ inch $= 0.2286$ m
$\rho = 1.71 \times 10^{-8}$ ohm meter (wall material: copper alloy 101)
$\mu_{r_{wall}} = 0.999991$

Since the exact solution assumes spherical ends, while NASA's truncated cone experiment has flat ends, the spherical radii r_1 and r_2 are calculated as the mean value of the radii to a) the intersection of the ends with the lateral conical walls and b) the top of the dome. From analysis of the problem and verification using numerical analysis (comparison with COMSOL FEA solutions for a large number of examples) I have determined that this mean value is an excellent approximation to the solution of Maxwell's equations for a truncated cone with flat ends. These input parameters result in the following values (in SI units) for the spherical radii r_1 and r_2:

$r_1 = 0.305316$ m

$r_2 = 0.537845$ m

and for the truncated cone half angle value at the conical wall θ_w (the spherical polar angle measured from the axis of symmetry z of the cone to the conical wall):

$\theta_w = 14.8125$ degrees

Experimental measurement by NASA (for mode shape TE012):

$f_{012} = 2.168$ GHz

Output (exact solution output results for mode shape TE012):

$f_{012} = 2.16467$ GHz
$\delta = 1.41457$ micrometers
$Q = 78,642.4$

Scaled geometry: ten times larger than NASA's geometry

Input

$D_b = 110.1$ inch $= 2.79654$ m
$D_s = 62.5$ inch $= 1.5875$ m
$L = 90.0$ inch $= 2.286$ m
$\rho = 1.71 \times 10^{-8}$ ohm meter (wall material: copper alloy 101)
$\mu_{r\,wall} = 0.999991$

Output (exact solution results for mode shape TE012):

$f_{012} = 0.216467$ GHz
$\delta = 4.43121$ micrometers
$Q = 251,049$

frequency scaling: $(2.16467231443342628^9/2.16467231443342667^8)/10 = 1$
Q scaling: $(78642.44767279371/251049.34868706256)/\sqrt{10} = 0.990599$

Scaled geometry: ten times smaller than NASA's geometry

Input

$D_b = 1.101$ inch $= 0.0279654$ m
$D_s = 0.625$ inch $= 0.015875$ m
$L = 0.900$ inch $= 0.02286$ m
$\rho = 1.71 \times 10^{-8}$ ohm meter (wall material: copper alloy 101)
$\mu_{r\,wall} = 0.999991$

Output (exact solution output results for mode shape TE012):

$f_{012} = 21.6467$ GHz

$\delta = 0.443121$ micrometers
$Q = 25,104.9$

frequency scaling: $(2.1646723144342628^9/2.164672314434267^{10}) * 10 = 1$
Q scaling: $(78642.44767279371/25104.934868706456)/\sqrt{10} = 0.990599$

The following is confirmed: when using the exact solution for resonance of a truncated conical cavity, for constant resistivity and magnetic permeability of the interior wall of the cavity and for constant geometrical ratios, constant medium properties and for the same mode shape (TE012): 1. the frequency scales (exactly) like the inverse of any geometrical dimension, 2. therefore the skin depth scales (exactly) like the square root of any geometrical dimension, 3. the quality of resonance (Q) scales approximately like the square root of any geometrical dimension, within 1% accuracy due to the approximation that the electromagnetic energy density is approximately constant through the interior volume and through the interior surface area of the cavity (this approximation is good for a low mode like TE012 but is expected to gradually degrade with higher mode shape numbers).

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3. *Notsosureofit*, Ph.D. in General Relativity, is a physicist in the Boston area, who would like to remain anonymous using this pen name that expresses his views about the experimental claims made about the EM Drive. The mathematical basis for his hypothesis can be accessed at `http://emdrive.wiki/@notsosureofit_Hypothesis`

4. Numerical values, to 15 digits precision, of the n^{th} roots x_{mn} and x'_{mn} of the cylindrical Bessel function $J_m(x)$ and its derivative $J'_m(x)$, respectively, for the following values of m and n: $m < 11$ and $n < 6$, available at `http://wwwal.kuicr.kyoto-u.ac.jp/www/accelerator/a4/besselroot.htmlx`

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